Final Exam Probability Theory (WIKR-06)

21 June 2018, 09.00-12.00

- Every exercise needs to be handed in on separate sheets, which will be collected in separate piles.
- Write your name and student number **on every sheet**.
- It is absolutely not allowed to use calculators, phones, the book, notes or any other aids.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes you are using (unless it is stated explicitly in the question that this is not needed.).
- **NOTA BENE:** using separate sheets for the different exercises and writing your name and student number on all sheets is worth 10 out of the 100 points.

Exercise 1, (a:6, b:6, c:6, d:6, e:6 pts).
The random variables X en Y have joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{(x+y)^3} & \text{if } x > c \text{ and } y > c, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $c \geq 0$ is a constant to be specified.)

(a) Show that, for all $a, b \geq c$:

$$\mathbb{P}(X > a \text{ and } Y > b) = \frac{1}{2(a+b)}.$$

(b) Show that the cdf of X is given by:

$$F_X(x) = \begin{cases} 1 - \frac{1}{2(x+c)} & \text{if } x \geq c, \\ 0 & \text{otherwise.} \end{cases}$$

(c) What is the value of c?

(d) Are X and Y dependent or independent?
(Motivate your answer.)

(e) What is $\mathbb{E}X$?
(Hint: an expectation is not always a real number, but it can also be $-\infty, +\infty$ or "undefined". Also, you may want to use $\frac{x}{(x+c)^2} = \frac{1}{x+c} - \frac{c}{(x+c)^2}$.)

Exercise 2, (a:5, b:5, c:5, d:5, e:5, f:5 pts).
A *tontine* is a kind of precursor to the pension, invented in 1653 by Lorenzo de Tonti. It works as follows: the participants of the tontine each pay a fee of a at the start. Every year thereafter a fixed total amount b ("dividend") is distributed evenly among the participants that are still alive. So if in a given year there are 100 surviving participants then each of them gets $b/100$ that year. When all participants have died nothing is paid out.

Louis XIV for instance wrote out a tontine in 1689 to finance his military campaigns. The participation fee was 300 livres and the last surviving participant died 31 years later at 96 years of age. She was receiving 73000 livres every year at that time.

A king writes out a tontine with initial participation fee $a = 40$ ducats. He sends his soldiers to convince, with force if necessary, the parents of all n newborns in his kingdom to take part. The dividend will be $b = n$ ducats every year. Let us assume the lifetimes of the newborns can be modelled by random variables $X_1, \dots, X_n$ that are i.i.d. exponential with parameter $\beta = 30$. So the expected lifetime is 30 (years).

(a) Give the pdf of X_1.
(The answer suffices here.)

The king ends up with a loss if at least one of the newborns will reach the ripe old age of 40. His minister of probabilistic matters advises him to study the distribution of the quantity $M := \max(X_1, \dots, X_n)$.

(b) Show that the cdf of M equals $F_M(x) = (1 - e^{-x/\beta})^n$, and also determine its pdf f_M.

Because the payment of dividends takes place every year, the random variables $Y_i := \lceil X_i \rceil$ contain all the information that is relevant for the king. (The notation $\lceil x \rceil$ means "round x upwards to the nearest integer". For convenience we assume all newborns were born on 1 January at 00:00:01 and the payment takes place on 31 December at 23:59:59.)

(c) Show that Y_i has a geometric distribution and determine its parameter p.

The first year when the king won't have to pay out dividend is $M' := \max(Y_1, \dots, Y_n)$

(d) Show that $\mathbb{P}(M' \leq k) = (1 - e^{-k/\beta})^n$ for all $k \in \mathbb{N}$.
(Hint: use part (b).)

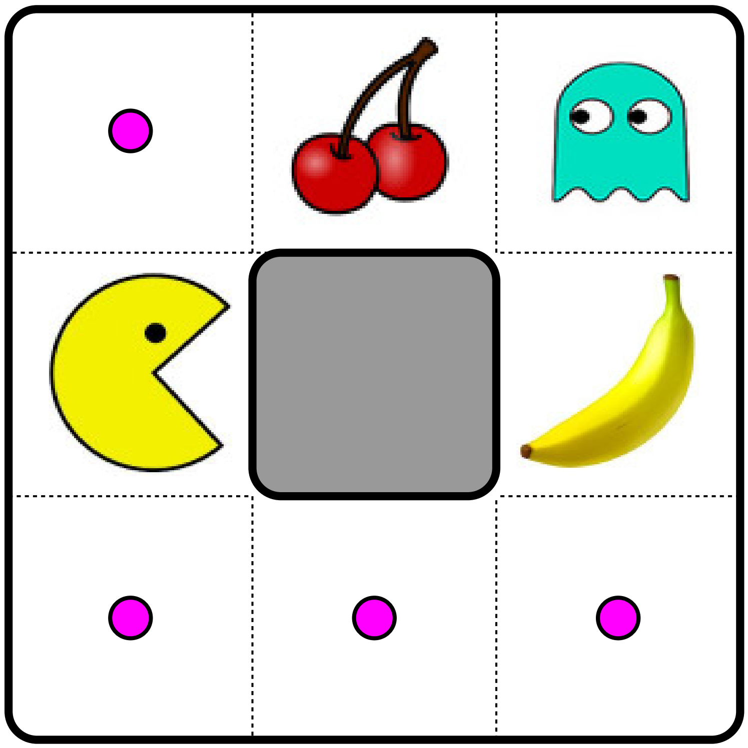
Only $n = 2$ families have entered their newborn into the tontine (either because there were an unusually low number of births this year, or because the soldiers have ignored the king's commands).

(e) Show that in this case, for all $k \in \mathbb{N}$:

$$\mathbb{P}(M' = k) = 2e^{-(k-1)/\beta} - 2e^{-k/\beta} - e^{-2(k-1)/\beta} + e^{-2k/\beta}.$$

(f) What is, still assuming $n = 2$, the expected profit of the king?
(Since the king has access to much better medical care, we can assume he'll outlive these two subjects even though he is substantially older.)
(**Note:** You may use without proof that for all $-1 < x < 1$ we have: $\sum_{k=1}^{\infty} kx^{k-1} = 1/(1-x)^2$.)

Exercise 3, (a: 10, b: 10, c: 10 pts)
A collector buys an old game console at a flea market with the game pacman on it. In the figure below you see the first "level" of the game. The playing field consists of 8 squares (in the middle there is an area where pacman cannot go), there are four pills, a banana, a cherry and a ghost.



If pacman ends up on the same square as the ghost the game is over (in this, very old, version of the game he has only one life). The goal of the game is to collect the pills, the banana and the cherry without coming in contact with the ghost. Because this is the first, very easy, level of the game, the ghost always stays on the same square.

The collector tries to play the game, but unfortunately something is wrong with the wiring of the game console. Because of this pacman always moves to a randomly chosen neighbouring square of the square he's currently in.

(a) In the first instance, we're interested in the probability pacman manages to reach the banana without coming in contact with the ghost. Write down a system of equations that describes this probability.

(b) Solve this system. What is the sought probability?

(c) What is the probability that pacman will first eat the banana and then the cherry, without coming in contact with the ghost?